JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 2, March–April 1994

Gyroscope Free Strapdown Inertial Measurement Unit by Six Linear Accelerometers

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A six-accelerometer configuration is presented to compute the rotational and translational acceleration of a rigid body. This theoretical minimum accelerometer configuration has as stable a mechanization equation as that of recent results of nine-accelerometer schemes. Associated equations that can be used to work with the accelerometer location and orientation errors are also derived. For navigation application, the novel design can be integrated with other sensors of complementary characteristics to enhance the performance.

Introduction

MOST current inertial navigation systems use linear accelerometers to sense linear accelerations and gyroscopes to sense angular velocity or angular position. However, it is possible to determine the kinematics of a rigid body by using only linear accelerometers. This viewpoint has been vigorously studied for over 20 years in connection with a number of diverse disciplines, such as inertial navigation, experimental modal analysis, and biomechanical research.[1-3] In theory, a minimum of six linear accelerometers are required for a complete description of a rigid body motion. However six-accelerometer schemes were unsuccessful in the past. Recent results show that workable schemes for a stable solution need nine accelerometers.[2-5] The key to the solution of this problem is the choice of location and orientation of these accelerometers.

We present a theoretical minimum workable six-accelerometer configuration. Our design has one accelerometer at the center of each face of a cube. The sensing axis of each accelerometer is along the respective cube face diagonal, in such a way that these diagonals form a regular tetrahedron. The rotational acceleration of the body can be computed from six acceleration field measurements. Direct integration of $\dot{\omega}$ with given initial condition $\omega(0)$ yields angular velocity ω. The translational acceleration at the center of the configuration is computed from the measurements and the knowledge of angular velocity.

Basic Principle

The accelerometer is a precision instrument designed to measure a component of the vector quantity called specific force f. It is defined by the equation

$$f = a - g \tag{1}$$

where a is inertial acceleration and g is gravitational attraction per unit mass. In navigation computation, we must calculate g and use $f + g$ to obtain the inertial acceleration a of a moving vehicle. For simplicity and without loss of generality, we will neglect the gravitation effect in the following discussion.

In describing the motion of a vehicular system, consider an inertial frame (I frame) and a rotating moving frame (b frame) as shown in Fig. 1. The acceleration of point P is given by

$$a = (\ddot{R})_I + (\ddot{r})_b + \dot{\omega} \times r + 2\omega \times (\dot{r})_b + \omega \times (\omega \times r) \tag{2}$$

where $(\ddot{r})_b$ is the acceleration of point P relative to body frame. $(\ddot{R})_I$ is acceleration of O_2 relative to O_1. The term $2\omega \times (\dot{r})_b$ is known as the Coriolis acceleration, $\omega \times (\omega \times r)$ represents a centripetal acceleration, and $\dot{\omega} \times r$ is the tangential acceleration owing to angular acceleration of the rotating frame.

If P is fixed in the b frame, the terms $(\dot{r}_i)_b$ and $(\ddot{r}_i)_b$ vanish. Thus a body rigidly mounted accelerometer at location r_i with sensing direction $\hat{\theta}_i$ would produce output of A_i

$$A_i = \left[(\ddot{R})_I + \dot{\Omega} r_i + \Omega\Omega r_i\right] \cdot \hat{\theta}_i \tag{3}$$

where

$$\Omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{4}$$

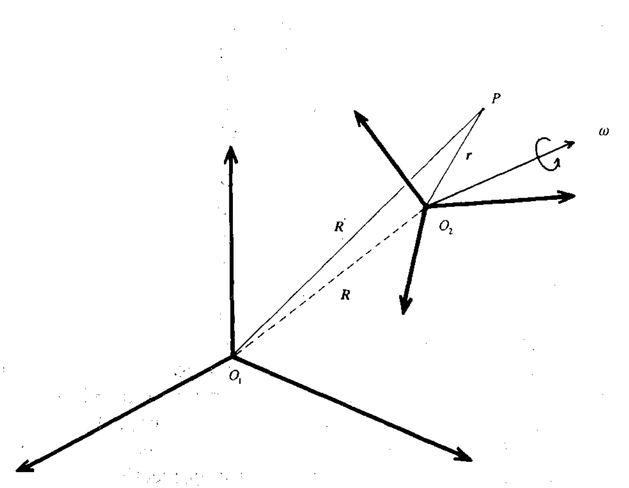

Fig. 1 Geometry of moving frame (b) and inertial frame (I).

Received Nov. 5, 1992; revision received April 4, 1993; accepted for publication May 9, 1993.

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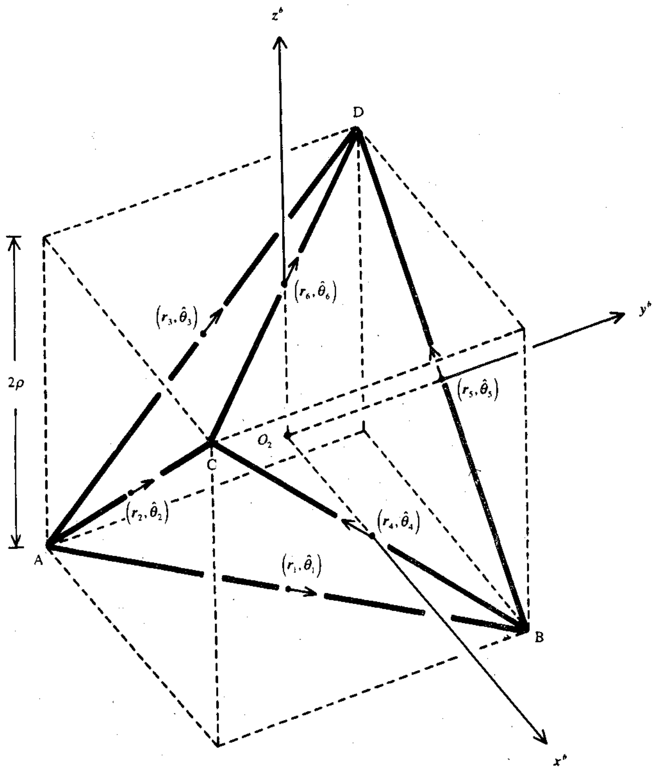

Fig. 2 Six-accelerometer configuration.

Configuration Description

The six-accelerometer configuration is shown in Fig. 2. Point O_2, the center of geometry, is defined as the origin of the moving frame. The accelerometer locations r_i scaled by distance factor ρ and sensing directions $\hat{\theta}_i$, where $i=1,\dots,6$, are expressed by

$$[r_1, r_2, r_3, r_4, r_5, r_6] = \rho\begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$[\hat\theta_1, \hat\theta_2, \hat\theta_3, \hat\theta_4, \hat\theta_5, \hat\theta_6] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{6}$$

Note that the sensing axes along the respective cubic face diagonals form a regular tetrahedron. To illustrate the basic operation of this configuration, consider the x axis only (see Fig. 2). By the symmetrical property of this configuration, the angular acceleration $\dot{\omega}$ along this axis can be determined by the output sum of a particular set of the four sensors $A_1 - A_2 + A_5 - A_6$ (the minus sign indicates the negative of the sensor output), because all of the linear acceleration $(\ddot{R})_I$ and centripetal acceleration $\Omega\Omega r$ components cancel.

The angular velocity ω is obtained by integrating $\dot{\omega}$ if $\omega(0)$ is given. With the same set of accelerometers and changing the output sign at positions 2 and 5 (i.e., $A_1 + A_2 - A_5 - A_6$) the linear acceleration $(\ddot{R})_I$ along the x axis is determined as follows: First, the tangential acceleration $\dot{\Omega} r$ components cancel and the centripetal acceleration components remain. However, the centripetal acceleration can be computed from the knowledge of ω. Therefore the linear acceleration is obtained by summing the four sensor outputs with corrections of centripetal effect.

Mechanization Equation

By applying Eqs. (5) and (6) to Eq. (3) we have six accelerometer outputs A_{io}, $i=1,\dots,6$, as follows:

$$A_o = \left[\rho S^T \,\vdots\, T^T\right]\begin{bmatrix} \dot\omega \\ \cdots \\ (\ddot{R})_I \end{bmatrix} - \rho T^T \begin{bmatrix} \omega_y\omega_z \\ \omega_z\omega_x \\ \omega_x\omega_y \end{bmatrix} \tag{7}$$

where

$$A_o = [A_{1o}, A_{2o}, A_{3o}, A_{4o}, A_{5o}, A_{6o}]^T$$

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{8}$$

The ith column vectors in matrices S and T are $1/\rho\ (r \times \hat\theta_i)$ and $\hat\theta_i$, respectively. Inverting Eq. (7), we have angular acceleration and linear acceleration in the body (b) frame.

$$\begin{bmatrix} \dot\omega \\ \cdots \\ (\ddot{R})_I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1/\rho)S \\ \cdots \\ T \end{bmatrix}\begin{bmatrix} A_{1o} \\ A_{2o} \\ A_{3o} \\ A_{4o} \\ A_{5o} \\ A_{6o} \end{bmatrix} + \rho\begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_y\omega_z \\ \omega_z\omega_x \\ \omega_x\omega_y \end{bmatrix} \tag{9}$$

Using $\dot\omega^b$ and $(\ddot{R})_I^b$ as the inertially referenced measurements and perturbing the navigation equation, we have error equations in the inertial frame[6]

$$\delta\dot\omega^b = \nu_1 \tag{10}$$

$$\delta\dot\phi^I = -(C_b^I)_{\text{nom}}\delta\omega^b \tag{11}$$

$$\delta\dot{x}^I = \delta v^I \tag{12}$$

$$\delta\dot{v}^I = (C_b^I)_{\text{nom}}[E\,\delta\omega^b + \nu_2] + (\ddot{R})^I_{I_{\text{nom}}} \times \delta\phi^I \tag{13}$$

where

$$E = \begin{bmatrix} 0 & \omega_z & \omega_y \\ \omega_z & 0 & \omega_x \\ \omega_y & \omega_x & 0 \end{bmatrix}_{\text{nom}} \tag{14}$$

The physical vector coordinated in a particular reference frame is indicated by superscript. δx^I, δv^I, $\delta\phi^I$, and $\delta\omega^b$ are the error state of position, velocity, attitude, and angular rate, respectively. $(C_b^I)_{\text{nom}}$ and $(\ddot{R})^I_{I_{\text{nom}}}$ are the nominal direction cosine matrix and acceleration. The terms ν_1 and ν_2 are the grouped accelerometer noises that can be treated statistically. The effects of the error sources on the states are obvious and quite similar to the gyro-based system. Note that there is a need for the initial condition of $\omega(0)$, therefore the conventional gyrocompass cannot be applied to this system.

Orientation and Location Error Effect

When sensor location and orientation errors are considered, the sensor output (7) must be modified. In the following we describe the appropriate sensor output as well as the equations for computing $\dot{\omega}$ and $(\ddot{\boldsymbol{R}})_I$.

The orientation errors for an accelerometer, α_5 (out of plane) and β_5 (in-plane), are shown in Fig. 3. These small error angles (α_i, β_i) are defined as follows: By rotating the cubic face plane to a horizontal position with $\boldsymbol{r}_i$ up, then the sensor output $\boldsymbol{g}\alpha_i$ is an indication of the angle α_i. The angle β_i, which is perpendicular to α_i, is obtained similarly by rotating the ith nominal sensing axis $\hat{\theta}_i$ 90 deg.

When the orientation error has been resolved, we can proceed to determine the location error for the accelerometers. Let

$$\boldsymbol{r}_i^* = [r_x, r_y, r_z]_i^T + [\mu_i, \eta_i, \xi_i]^T$$
$$= \boldsymbol{r}_i + \Delta\boldsymbol{r}_i \tag{15}$$

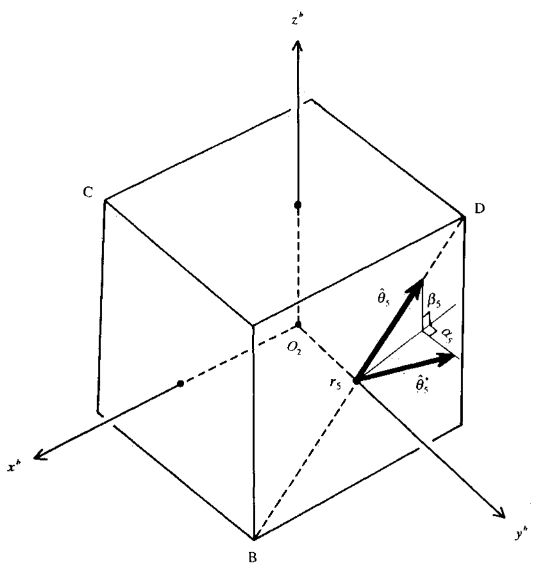

Fig. 3 Sensor orientation error (α_i, β_i).

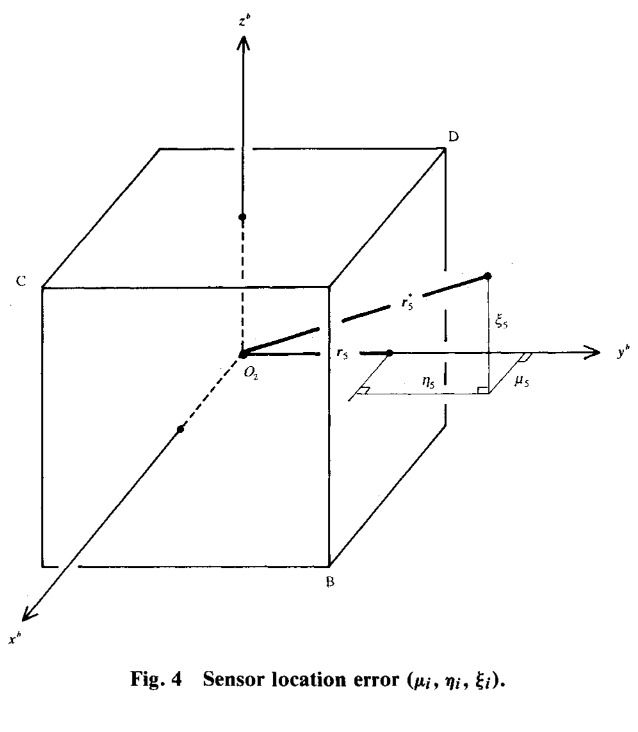

Fig. 4 Sensor location error (μ_i, η_i, ξ_i).

where $\boldsymbol{r}_i^*$ is the location of the ith accelerometer, and $\boldsymbol{r}_i$ and $\Delta\boldsymbol{r}_i$ represent nominal location and location error, respectively (see Fig. 4). In principle, the distance perpendicular to the spinning axis can be determined by the centripetal acceleration if the rotation rate is known. This concept could be applied to evaluate the sensor location errors. The following example illustrates the point. Because of the rotation Ω_1 (for simplicity, we neglect the gravitation effect) the difference between the sensor output (A_5^*) and the calculated value (A_5') is

$$A_5^* - A_5' = (\Omega_1\Omega_1\boldsymbol{r}_5^*)\cdot\hat{\theta}_5^* - (\Omega_1\Omega_1\boldsymbol{r}_5)\cdot\hat{\theta}_5^*$$
$$= (\Omega_1\Omega_1\hat{\theta}_5^*)\cdot\Delta\boldsymbol{r}_5 \tag{16}$$

By defining

$$a_i = \cos(\alpha_i)$$
$$b_i = \cos(\pi/4 - \beta_i)$$
$$c_i = \sin(\alpha_i)$$
$$d_i = \sin(\pi/4 - \beta_i)$$

then (referring to Fig. 3)

$$\hat{\theta}_5^* = [-a_5 d_5, c_5, a_5 b_5]^T \tag{17}$$

Equation (16) can be rewritten as

$$\boldsymbol{w}_1 \cdot \Delta\boldsymbol{r}_5 = y_1 \tag{18}$$

where

$$\boldsymbol{w}_1 = \Omega_1\Omega_1\hat{\theta}_5^*$$
$$y_1 = A_5^* - A_5'$$

$\boldsymbol{w}_1$, y_1 are treated as known quantities. To do another two rotations Ω_2 and Ω_3, all three vector equations can be written in a matrix form as

$$\boldsymbol{W}\Delta\boldsymbol{r}_5 = \boldsymbol{y} \tag{19}$$

where

$$\boldsymbol{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3]^T$$
$$\boldsymbol{y} = [y_1, y_2, y_3]^T$$

When the three rotation axes are not in the same plane, the location errors have a unique solution. It is

$$\Delta\boldsymbol{r}_5 = \boldsymbol{W}^{-1}\boldsymbol{y} \tag{20}$$

The other five sensor locations error could be obtained in a similar way. Now the location and orientation errors of the six accelerometers can be expressed by $(\mu_i, \eta_i, \xi_i, \alpha_i, \beta_i)$, where $i = 1, \ldots, 6$. Similar to Eq. (3), the sensor output at position $\boldsymbol{r}_1^*$, including the effect of location and orientation errors, can be expressed as

$$A_1^* = \left[(\ddot{\boldsymbol{R}})_I + \dot{\Omega}\boldsymbol{r}_1^* + \Omega\Omega\boldsymbol{r}_1^*\right]\cdot\hat{\theta}_1^* \tag{21}$$

where

$$\hat{\theta}_1^* = [a_1 b_1, a_1 d_1, -c_1]^T$$
$$\boldsymbol{r}_1^* = \boldsymbol{r}_1 + \Delta\boldsymbol{r}_1$$

Equation (21) can be expressed in matrix form as

$$A_1^* = \left\{ \begin{bmatrix} (\ddot{R}_x)_I & -(\omega_y^2+\omega_z^2) & -\dot{\omega}_z+\omega_x\omega_y & -(\dot{\omega}_y+\omega_z\omega_x) \\ (\ddot{R}_y)_I & \dot{\omega}_z+\omega_x\omega_y & -(\omega_y^2+\omega_z^2) & \dot{\omega}_x-\omega_y\omega_z \\ (\ddot{R}_z)_I & -\dot{\omega}_y+\omega_z\omega_x & \dot{\omega}_x+\omega_y\omega_z & \omega_x^2+\omega_y^2 \end{bmatrix} \begin{bmatrix} 1 \\ \mu_1 \\ \eta_1 \\ \rho-\xi_1 \end{bmatrix} \right\} \cdot \begin{bmatrix} a_1 b_1 \\ a_1 d_1 \\ -c_1 \end{bmatrix} \tag{22}$$

Applying the same manipulation for the other five accelerometer outputs gives, in matrix form,

$$A^* = M \begin{bmatrix} \dot{\omega} \\ \cdots \\ (\ddot{R})_I \end{bmatrix} + NV \tag{23}$$

where

$$A^* = \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ A_4^* \\ A_5^* \\ A_6^* \end{bmatrix}, \qquad V = \begin{bmatrix} \omega_y\omega_z \\ \omega_z\omega_x \\ \omega_x\omega_y \\ (\omega_x^2+\omega_y^2) \\ (\omega_z^2+\omega_x^2) \\ (\omega_y^2+\omega_z^2) \end{bmatrix} \tag{24}$$

$$M = \begin{bmatrix} (\rho-\xi_1)a_1d_1-\eta_1c_1 & -(\rho-\xi_1)a_1b_1+\mu_1c_1 & -\eta_1a_1b_1+\mu_1a_1d_1 & a_1b_1 & a_1d_1 & -c_1 \\ \xi_2c_2-(\rho-\eta_2)a_2b_2 & \xi_2a_2d_2-\mu_2a_2b_2 & (\rho-\eta_2)a_2d_2-\mu_2c_2 & a_2d_2 & -c_2 & a_2b_2 \\ -\xi_3a_3b_3+\eta_3a_3d_3 & -\xi_3c_3+(\rho-\mu_3)a_3d_3 & \eta_3c_3-(\rho-\mu_3)a_3b_3 & -c_3 & a_3b_3 & a_3d_3 \\ \xi_4a_4b_4+\eta_4a_4d_4 & \xi_4c_4-(\rho+\mu_4)a_4d_4 & -\eta_4c_4-(\rho+\mu_4)a_4b_4 & c_4 & -a_4b_4 & a_4d_4 \\ -\xi_5c_5+(\rho+\eta_5)a_5b_5 & -\xi_5a_5d_5-\mu_5a_5b_5 & (\rho+\eta_5)a_5d_5+\mu_5c_5 & -a_5d_5 & c_5 & a_5b_5 \\ -(\rho+\xi_6)a_6d_6+\eta_6c_6 & -(\rho+\xi_6)a_6b_6-\mu_6c_6 & \eta_6a_6b_6+\mu_6a_6d_6 & -a_6b_6 & a_6d_6 & c_6 \end{bmatrix} \tag{25}$$

$$N = \begin{bmatrix} -\eta_1c_1-(\rho-\xi_1)a_1d_1 & -\mu_1c_1-(\rho-\xi_1)a_1b_1 & \mu_1a_1d_1+\eta_1a_1b_1 & -(\rho-\xi_1)c_1 & -\eta_1a_1d_1 & -\mu_1a_1b_1 \\ -\xi_2c_2-(\rho-\eta_2)a_2b_2 & \xi_2a_2d_2+\mu_2a_2b_2 & -(\rho-\eta_2)a_2d_2-\mu_2c_2 & -\xi_2a_2b_2 & -(\rho-\eta_2)c_2 & -\mu_2a_2d_2 \\ \xi_3a_3b_3+\eta_3a_3d_3 & -\xi_3c_3-(\rho-\mu_3)a_3d_3 & -\eta_3c_3-(\rho-\mu_3)a_3b_3 & -\xi_3a_3d_3 & -\eta_3a_3b_3 & -(\rho-\mu_3)c_3 \\ -\xi_4a_4b_4+\eta_4a_4d_4 & \xi_4c_4+(\rho+\mu_4)a_4d_4 & \eta_4c_4-(\rho+\mu_4)a_4b_4 & -\xi_4a_4d_4 & \eta_4a_4b_4 & -(\rho+\mu_4)c_4 \\ \xi_5c_5+(\rho+\eta_5)a_5b_5 & -\xi_5a_5d_5+\mu_5a_5b_5 & -(\rho+\eta_5)a_5d_5+\mu_5c_5 & -\xi_5a_5b_5 & -(\rho+\eta_5)c_5 & \mu_5a_5d_5 \\ (\rho+\xi_6)a_6d_6+\eta_6c_6 & -(\rho+\xi_6)a_6b_6+\mu_6c_6 & -\eta_6a_6b_6+\mu_6a_6d_6 & -(\rho+\xi_6)c_6 & -\eta_6a_6d_6 & \mu_6a_6b_6 \end{bmatrix} \tag{26}$$

The rotational and translational acceleration of the moving frame are calculated by

$$\begin{bmatrix} \dot{\omega} \\ \cdots \\ (\ddot{R})_I \end{bmatrix} = M^{-1}(A^* - NV) \tag{27}$$

Comparing this to Eq. (9), the orientation and location error terms introduce the nonlinear effect of angular velocity to the $\dot{\omega}$ equation. This may cause the instability in angular velocity computation to first order of the error quantities.[4]

Sensor Error Effect

The sensor errors can be calibrated by a general testing procedure in the laboratory. Assume that after the calibration there is still a small residual error (for example a 10 μg bias error). A cubic edge length of 20 cm mechanization will cause 56.2 mdeg/s^2 in angular acceleration error. Therefore, the errors of angular velocity ($\delta\omega$) and angular position ($\delta\theta$) grow with t and t^2, respectively. For a static and flat land case, it produces a linear acceleration error of ($g\delta\theta$), which results in a velocity and position error proportional to t^3 and t^4, respectively [see Eqs. (10–13)]. The attitude and position errors produced by a 56.2 mdeg/s^2 angular acceleration bias for a time of 10 s are 2.81 deg and 4 m, respectively.

The errors and error buildup rates are atrocious compared with the medium-accuracy gyro-based system. However, the system is substantially rugged and has the advantages of no wear out, low power consumption, fast reaction, and high angular acceleration measurement capability. Its immediate applications in pure inertial mode would cover short mission duration systems with extremely high angular dynamic environments, or it can be used as the rotational-translational sensor. For relatively long mission time application, an estimator with measurements from external sensors of complementary error characteristics (e.g., global positioning system) can bound the navigation errors. The extensive studies of the aided inertial navigation system in the past can be applied to this system.

Conclusion

This paper has developed a six-accelerometer strapdown inertial measurement unit. The basic operation of this configuration is illustrated in detail. The mechanization equation for navigation is computationally stable. Since it is an acceleration-type (both linear and angular) measurement unit, the angular rate error grows with the mission time due to the sensor bias. The most interesting application of this mechanization with state-of-the-art fabrication techniques of accelerometer would cover short mission duration navigation systems with the advantages of low cost, fast reaction, and high angular acceleration capability. For relatively long mission time applications, it can be integrated with other sensors of complementary characteristics to enhance the performance. This rotational-translational sensor can also be used as a more accurate vibration monitoring device for industrial or biomechanical research.

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